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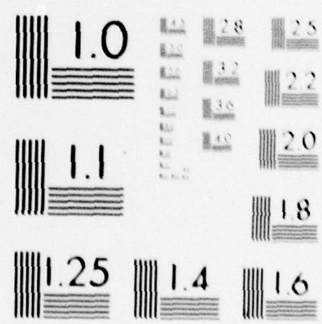

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Matched and Wiener filters are considered for signal processing applications when the a priori information about signal and noise characteristics are not completely specified. The approach is to design filters which are saddle-point or max-min solutions for the criterion functional (mean-squared-error or signal-to-noise ratio) over the classes of allowable signal shapes and signal and noise spectral densities. Two-dimensional discrete-parameter processes are considered, and some numerical examples are presented.

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SIGNAL PROCESSING FILTERS UNDER MODELING UNCERTAINTIES

Saleem A. Kassam, Tong Leong Lim and Leonard J. Cimini

ABSTRACT

Matched and Wiener filters are considered for signal processing applications when the *a priori* information about signal and noise characteristics are not completely specified. The approach is to design filters which are saddle-point or max-min solutions for the criterion functional (mean-squared-error or signal-to-noise ratio) over the classes of allowable signal shapes and signal and noise spectral densities. Two-dimensional discrete-parameter processes are considered, and some numerical examples are presented.

I. INTRODUCTION

Classical formulations of signal processing problems assume that the characteristics of signals and noise can be modeled exactly, either deterministically or statistically. For example, if the shape of a deterministic signal is known and the noise additively corrupting it has a known power spectral density (PSD), then a filter maximizing the signal-to-noise ratio (SNR) at its output can be designed; this results in the well-known matched filter. In the same way, the optimum Wiener filter can be obtained for the best linear estimate of a random signal in additive noise, when both PSD's are known.

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In many applications it is much more reasonable to assume that the signal and noise characteristics are not completely known, and to assume that they can only be defined as belonging to appropriate classes of characteristics. The sizes of these classes reflect the degrees of uncertainty that one has about the true signal and noise characteristics. In such cases, it is desirable to have filters which perform well over both classes of allowable signal and noise characteristics, that is, we should look for robust filters.

In general the specification of an optimum filter for processing inputs requires knowledge of the multivariate probability distribution functions characterizing the input random processes. For estimation and detection applications under the widespread assumption of Gaussian input processes the optimum filters are generally linear, and are based on the bivariate density functions, that is, the mean and covariance functions, of the input Gaussian process. Even if the input processes are not Gaussian, a restriction to linear filtering, as in Wiener and matched filtering, allows optimum filters to be obtained if mean and covariance function information is available. In this paper we will be dealing with cases of linear filtering where this bivariate information is imprecise. Other efforts in robustness theory have been concerned with the deviations from Gaussian distributions which may occur in input processes. Allowances for distributional impreciseness have led to many interesting results on robust nonlinear structures in detection and estimation theory [e.g., 1-5]; however, such results deal almost exclusively with univariate density functions and hence "white" inputs because of the major analytical difficulties which otherwise appear.

One of the earliest investigations of robust linear filtering ideas was reported by Yovits and Jackson in 1955 [6]. They considered a game-theoretic problem of max-min filter design for signal estimation in additive noise with mean-squared-error as the pay-off functional, the signal having constraints on its mean-square derivatives. Although the realizable Wiener filter was considered, only for the white-noise case was it possible to obtain explicit results. In [7] and [8] Nahi and Weiss obtained "bounding filters" for the Wiener and Kalman filter, which were possibly lower-order filters with guaranteed error performance for classes of input characteristics. More recently, Kuznetsov [9] obtained the saddle-point matched filter for the SNR criterion; specifically, with the Fourier transform $S(\omega)$ of a finite energy signal allowed to be a member of a class C_s, and for the noise spectral density $N(\omega)$ allowed to be a member of a class C_n, the saddle-point solutions were obtained for the game with pay-off SNR(S,N;H). Here SNR(S,N;H) is the SNR obtained when filter H is used for signal and noise characteristics $S(\omega)$, $N(\omega)$, respectively. Kuznetsov obtained separately the robust filter for $S(\omega)$ in C_s with $N(\omega)$ known, and for $S(\omega)$ known and $N(\omega)$ in C_n. The class C_s was the class of finite-energy signals which are within an allowable distance $\Delta > 0$ of a nominal $S_0(\omega)$ characteristic, in the sense of L_2 distance. The class C_n was the "band-model" class of the spectral densities bounded by given densities $N_L(\omega) \le N_U(\omega)$, with a total-power constraint.

In [10], Kassam and Lim obtained the structures of robust (saddle-point) filters for the Wiener filter formulation, with both signal and noise spectral densities allowed to be members of classes of densities specified by "band-models". In [11], Poor generalizes some results on Wiener filters obtained in [10].

Here we will obtain the robust matched and Wiener filters for two-dimensional discrete-parameter systems subject to uncertainties simultaneously about signal and noise characteristics. Thus this work represents a logical extension of the recent results in [9-11].

II. ROBUST MATCHED FILTERES

Let s(m,n) be a real, finite-energy, two-dimensional, deterministic signal sequence, where m,n are integer variables, and let S(u,v) be the two-dimensional discrete-time Fourier transform of s(m,n). If N(u,v) is the PSD of a real, stationary, zero-mean, additive noise, then the matched filter with frequency response

$$H^{\dagger}(u,v) = S^{*}(u,v)/N(u,v) \tag{1}$$

maximizes the SNR at its output at m=0, n=0. If an arbitrary filter H(u,v) is used, the SNR functional is

$$SNR(S,N;H) = \left(\frac{1}{2\pi}\right)^2 \frac{\left|\iint S(u,v)H(u,v)dudv\right|^2}{\iint N(u,v)\left|H(u,v)\right|^2 dudv} \tag{2}$$

In (2) and the rest of this paper, except where indicated otherwise, every double integral is over the region $\{u,v|-\pi<u,v\le\pi\}$. In the following, we will drop the arguments of the functions wherever no confusion results.

Consider the following models for the classes of allowable signal and noise characteristics:

$$C_s = \{S(u,v) \mid (\tfrac{1}{2\pi})^2\iint|S-S_0|^2 \; dudv \le\delta\}, \tag{3}$$

$$C_n = \{N(u,v) \mid N_L(u,v)\le N(u,v)\le N_U(u,v) \text{ and } (\tfrac{1}{2\pi})^2\iint N(u,v) \; dudv = \sigma_N^2\} \tag{4}$$

In the definition of C_s, $S_0(u,v)$ is a nominal signal characteristic, and signals in C_s differ in energy from the nominal signal $s_0(m,n)$ by no more than δ. To exclude a trivial case, we assume that the energy $\sum_m\sum_n|s_0(m,n)|^2>\delta$. The class C_s is a reasonable model for the allowable signal when it is known that s(m,n) is within a neigborhood of $s_0(m,n)$. The class C_n of noise PSD's contains PSD's with a specific total power σ_N^2 which lie between given upper and lower bounds. Such a model is appropriate, for example, when N(u,v) is estimated from samples. In the definition of C_n, we assume that $N_L(u,v)$ is bounded.

The robust matched filter $H_R(u,v)$ is defined to be the max-min filter such that

$$\max_{H} \min_{\substack{S\in C_s \\ N\in C_n}} SNR(S,N;H) = \min_{\substack{S\in C_s \\ N\in C_n}} SNR(S,N;H_R) \tag{5}$$

If H_R is the optimum filter for a least-favorable pair of characteristics $S_R \epsilon C_s$, $N_R \epsilon C_n$, then we would have

$$SNR(S,N;H_R) \geq SNR(S_R,N_R;H_R) \geq SNR(S_R,N_R;H) \tag{6}$$

for any pair (S,N) in $C_s \times C_n$ and any linear filter H, and (5) would be satisfied. In this case, the filter $H=H_R$ and the pair $(S_R,N_R) \epsilon C_s \times C_n$ will form a saddle-point for the SNR functional of (2).

The main result of this section is the following theorem, which gives the saddle-point solution for our matched filtering problem:

<u>Theorem 1</u>. For the classes C_s and C_n, the robust matched filter H_R is the optimum filter for the pair of least favorable characteristics defined by

$$S_R = \frac{S_0(u,v)N_R(u,v)}{N_R(u,v) + c} \tag{7}$$

and

$$N_R = \begin{cases} N_U(u,v) & , \ (u,v)\epsilon R_U(k) \\ N_L(u,v) & , \ (u,v)\epsilon R_L(k) \\ |S_R(u,v)|/k & , \ (u,v)\epsilon R_M(k) \end{cases} , \tag{8}$$

where

$$R_U(k) = \{(u,v) \big| kN_U < |S_R|\} \quad , \tag{9}$$

$$R_L(k) = \{(u,v) \big| |S_R| \leq kN_L\} \quad , \tag{10}$$

$$R_M(k) = \{(u,v) \big| kN_L < |S_R| \leq kN_U\} \ , \tag{11}$$

if non-negative constants k,c exist satisfying

$$\frac{c^2}{(2\pi)^2} \iint \frac{|S_0|^2}{[N_R+c]^2} \, dudv = \delta \ , \tag{12}$$

and

$$\iint\limits_{R_L(k)} N_L \, dudv + \iint\limits_{R_U(k)} N_U \, dudv + \frac{1}{k} \iint\limits_{R_M(k)} |S_R| \, dudv = (2\pi)^2 \sigma_N^2 \qquad (13)$$

We omit the detailed proof of the theorem, since it is somewhat lengthy. However, some clarifying comments can be made, and an outline of the proof follows.

The structure of the solution is suggested by the results given in [9,10]. Eq. (12) results from conditions imposed in defining C_s, and (13) is the power constraint for the class C_n. Note that (7) and (8) are coupled, but that S_R and N_R can be expressed in terms of the known S_0, N_L, and N_U, and constants k and c, which can be solved from two simultaneous equations derived from (12) and (13). The proof shows when such a solution for k and c will exist. The saddle-point condition (6) can be verified by noting that H_R is optimum for S_R and N_R [one part of (6)], and by showing that (S_R,H_R) minimizes the numerator in (2) over all (S,H_R) with $S\varepsilon C_s$, (N_R,H_R) maximizes the denominator in (2) over all the (N,H_R) with $N\varepsilon C_n$.

For the case where either $\delta=0$ or $N_L = N_U$ the above results become somewhat simpler. Theorem 1 also gives, as a special case, the robust filter for a different model, the ε-model [10], for $N(u,v)$. We will consider this model in a simple example in the next section.

Outline of Proof of Theorem 1

Since H_R is the optimum filter for the pair (S_R, N_R), we have to show only that $SNR(S,N;H_R) \geq SNR(S_R,N_R;H_R)$ for all (S,N) in $C_s \times C_n$.

Consider the denominator term in (2), and call it $d(N,H)$. Now

$$d(N,H_R)-d(N_R,H_R)=k^2 \iint_{R_M(k)} (N-N_R)+\iint_{R_L(k)} (N-N_L)|H_R|^2+\iint_{R_U(k)} (N-N_U)|H_R|^2 \qquad (14)$$

On $R_L(k)$, $|H_R|^2 \leq k^2$ and on $R_U(k)$, $|H_R|^2 \geq k^2$; thus using the power constraint, we find $d(N,H_R)-d(N_R,H_R) \leq 0$.

In showing that the numerator term in (2) is minimized for $S=S_R$ given that $H=H_R$, we note that $\iint[S_0^* S/(N_R+c)]$ is non-negative, and we consider the minimization of $\iint[S_0^*(S-S_0)/(N_R+c)]$. This last term is real, so we consider

$$\iint[|S_0||S-S_0|/(N_R+c)]\cos[\arg S_0^*(S-S_0)]dudv$$

This term is a minimum when the cosine part is -1, and the result of (7) for the minimizing S can be obtained from Schwarz's inequality.

In order to show when the solution given by (7) and (8) exists, that is, when non-negative constants c and k exist which are solutions to (12) and (13), we rewrite Eqs. (12) and (13) as

$$\iint_{R_L(k,c)} N_L + \iint_{R_U(k,c)} N_U + \iint_{R_M(k,c)} \left(\frac{|S_0|}{k} - c\right) = (2\pi)^2\sigma_N^2 \qquad (15)$$

and

$$c^2\iint_{R_L(k,c)} \frac{|S_0|^2}{[N_L+c]^2} + c^2\iint_{R_U(k,c)} \frac{|S_0|^2}{[N_U+c]^2} + c^2\iint_{R_M(k,c)} k^2 = (2\pi)^2\delta \qquad (16)$$

where $R_L(k)$ of Eq. (10) has been written explicitly as

$$R_L(k,c) = \{(u,v) \,\big|\, |S_0| N_L \leq kN_L(N_L+c)\} , \tag{17}$$

and similarly

$$R_U(k,c) = \{(u,v) \,\big|\, kN_U(N_U+c) < |S_0| N_U \text{ and } N_L > 0\} , \tag{18}$$

$$R_M(k,c) = \overline{R_L(k,c)} \cap \overline{R_U(k,c)} \tag{19}$$

Suppose the condition

$$\iint\limits_{N_L=0} |S_0|^2 dudv < (2\pi)^2\delta < \iint\limits_{N_L=0} |S_0|^2 dudv + \iint\limits_{\substack{N_L>0 \\ N_U<\infty}} |S_0|^2 dudv \tag{20}$$

is satisfied. This will be true in many models of interest; it is valid, for example, when N_L and N_U are finite and non-zero when $|S_0|$ is positive. Then there exists a finite, positive solution $c=c_1$ in Eq. (12) with N_L replacing N_R, and there exists also a finite, positive solution $c=c_2 \geq c_1$ in the equation

$$\iint\limits_{N_L=0} |S_0|^2 dudv + c^2 \iint\limits_{N_L>0} \frac{|S_0|^2}{(N_U+c)^2} dudv = (2\pi)^2\delta \tag{21}$$

If the set $\{(u,v) \,\big|\, (N_U-N_L)N_L|S_0|>0\}$ has positive measure, then $c_2>c_1$. Otherwise, $c_2=c_1$.

Consider Eq. (16). Let $k=k_s(c)$ be the solution for k with $c\varepsilon[c_1,c_2]$. For $c=c_1$, we define $k_s(c_1)$ as the <u>ess sup</u> over the set $\{(u,v) \,\big|\, N_L|S_0|>0\}$ of $|S_0|/(N_L+c_1)$, and for $c=c_2$ we define $k_s(c_2)$ similarly as the <u>ess inf</u>, over the same set, of $|S_0|/(N_U+c_2)$. The solution $k_s(c)$ is a smooth, non-increasing function of c on $[c_1,c_2]$.

Suppose the condition

$$(2\pi)^2\sigma_N^2 < \iint\limits_{|S_0|=0} N_L\,dudv + \iint\limits_{|S_0|N_L>0} N_U\,dudv \tag{22}$$

is satisfied. This is also generally true. Then the solution $k=k_n(c)$ with $c\epsilon[c_1,c_2]$ for Eq. (15) exists, and we have $k_s(c_2) < k_n(c_2) \le k_n(c_1) < k_s(c_1)$. (Note that (22) is not compatible with $c_1=c_2>0$, given that $\iint N_L dudv < (2\pi)^2\delta < \iint N_U dudv$). The solution $k_n(c)$ is also a smooth, non-increasing function of c on $[c_1,c_2]$. Thus, positive solutions k,c always exist, when (20) and (22) are satisfied, for the simultaneous equations (12) and (13).

If (20) is true but (22) is not, we may pick $N_R=N_U$ when $|S_0|N_L>0$, $N_R=0$ when $N_L=0$ and $|S_0|>0$, and N_R arbitrary otherwise, with S_R as in Eq. (7) and $c=c_2$. Similar special cases can be considered when (20) is not satisfied, for example when N_U is unbounded everywhere.

If the set $\{(u,v)\,|\,(N_U-N_L)N_L|S_0|>0\}$ has measure zero (for example, $N_L=N_U$) and $\delta>0$ we take $c=c_1=c_2$ (assuming the LHS inequality in (20) is valid, otherwise the solution is trivial) and $N_R=N_L$ when $|S_0|>0$, arbitrary otherwise. If $\delta=0$ and $\{(u,v)\,|\,(N_U-N_1)N_L|S_0|>0\}$ has positive measure, we can set $c=0$ and take $k=k_n(0)$ the finite, positive solution of Eq. (15), assuming (22) is true (otherwise the solution is trivial). We will consider another special case in the next section.

III. A NUMERICAL EXAMPLE

We will now consider a simple specific model for the signal and noise characteristics and derive some numerical results. Consider the case where N(u,v) and S(u,v) are circularly symmetric, so that we may express N(u,v) as N(r) and S(u,v) as S(r) where $r = \sqrt{u^2+v^2}$. We define a particular nominal signal Fourier transform by

$$S_0(r) = \begin{cases} 4.0 \, , & 0 \leq r \leq 2.5 \\ 0 \, , & r > 2.5 \end{cases} \tag{23}$$

In our first example, we will assume that we have precise information about the noise,

$$\begin{aligned} N(r) &= N_0(r) \\ &= N_L(r) \\ &= N_U(r) \\ &= \begin{cases} 2.5-r, & 0 \leq r \leq 2.5 \\ 0 \, , & r > 2.5 \end{cases} \end{aligned} \tag{24}$$

If we take δ=0.55, which represents an uncertainity of about 7% of the total signal energy, we can easily solve for the value c=0.1. Thus we have

$$S_R(r) = \begin{cases} \dfrac{4.0(2.5-r)}{2.6-r} \, , & 0 \leq r \leq 2.5 \\ 0 \, , & r > 2.5 \end{cases} \tag{25}$$

and the robust filter is

$$H_R(r) = \begin{cases} \dfrac{4.0}{2.6-r} \, , & 0 \leq r \leq 2.5 \\ 0 \, , & r > 2.5 \end{cases} \tag{26}$$

The effect of the δ-uncertainity in the signal definition thus prescribes a filter designed on the assumption of an added white-noise component of level 0.1 for the noise spectrum. This white-noise components results in a non-singular signal detection solution, whereas the "optimum" filter H_0 designed for the nominal signal S_0 and noise N results in a theoretically infinite $SNR(S_0,N;H_0)$. However, if H_0 is used when $S \neq S_0$, we could get very different results. For example, it turns out that $SNR(S_R,N;H_0) = -\infty$ dB, but with H_R we get $SNR(S_R,N;H_R) = 10$ dB, which forms the lower bound for performance. When H_R is used and the nominal signal is in effect, we get $SNR(S_0, N; H_R) = 14$ dB.

To extend this example, let us introduce distinct upper and lower bounds for N:

$$N_L(r) = (1-\varepsilon)\ N_0(r)\ , \tag{27}$$

$$N_U(r) = \ \infty\ , \tag{28}$$

with $\varepsilon = 0.1$ and noise variance σ_N^2 assumed known to be the variance of $N_0(r)$, the "nominal" noise spectrum. This model may be interpreted as expressing a 90% confidence in the validity of $N_0(r)$ for the noise spectrum, with an arbitrary noise component allowed otherwise. In order to obtain S_R and N_R, $c=c_1$ was first found as the solution of Eq. (12) with N_L replacing N_R. The correct c has to be larger than this. The solution $k=k_s(c)$ for $c > c_1$ in Eq. (16) was then computed numerically. Similarly, the solution $k=k_n(c)$ for $c > c_1$ was computed numerically for Eq. (15). The intersection of these two results gave the value of $c=0.23$ and $k=5.9$. These results give

$$N_R(r) = \begin{cases} 0.9(2.5-r) & , \quad 0 \le r \le 2.0 \\ 0.45 & , \quad 2.0 < r \le 2.5 \\ 0 & , \quad r > 2.5 \end{cases} \tag{29}$$

and

$$S_R(r) = \begin{cases} \dfrac{3.6(2.5-r)}{0.9(2.5-r) + 0.23} & , \quad 0 \le r \le 2.0 \\ 2.65 & , \quad 2.0 < r \le 2.5\ , \\ 0 & , \quad r > 2.5 \end{cases} \tag{30}$$

so that

$$H_R(r) = \begin{cases} \dfrac{4.0}{2.5-0.9r} & , \quad 0 \le r \le 2.0 \\ 5.9 & , \quad 2.0 < r \le 2.5 \\ 0 & , \quad r > 2.5 \end{cases} \tag{31}$$

Figure 1 shows a sketch of these functions. Numerical values for SNR's can also be computed for this case of signal and noise uncertainity; the robust filter lower bound on SNR will now be somewhat less than 10 dB, because of the additional noise uncertainity.

IV. ROBUST WIENER FILTERS

Let $S(u,v)$ now be the PSD of a real, wide-sense stationary random signal observed as a mixture with real, wide-sense stationary, zero-mean, additive noise with PSD $N(u,v)$. It is well-known that the optimum linear filter giving the minimum mean-squared-error (MSE) estimate of the signal is given by

$$H^{\dagger}(u,v) = \frac{S(u,v)}{S(u,v)+N(u,v)} \quad , \qquad (32)$$

without realizability constraints.

In general, if an arbitrary filter $H(u,v)$ is used, the MSE is given by

$$e(S,N;H) = \left(\frac{1}{2\pi}\right)^2 \iint [S(u,v)\,|1-H(u,v)|^2 + N(u,v)\,|H(u,v)|^2]\, dudv \qquad (33)$$

and the *minimum* MSE is given by

$$e^{\dagger}(S,N) = e(S,N;H^{\dagger}) = \left(\frac{1}{2\pi}\right)^2 \iint \frac{S(u,v)N(u,v)}{S(u,v)+N(u,v)}\, dudv \qquad (34)$$

We would like to consider a method for designing the estimating filter $H(u,v)$ when the PSD's of the signal and noise processes are not precisely known. The approach which will be followed is a direct extension of the one-dimensional case given in [10].

We now assume that both the signal and noise PSD's belong to upper-lower bounded classes of the form of (4). Thus we now have the allowable signal and noise PSD's to be members of classes D_s and D_n defined by

$$D_s = \{S(u,v) \,|\, S_L(u,v) \le S(u,v) \le S_U(u,v) \text{ and } \left(\frac{1}{2\pi}\right)^2 \iint S(u,v)\,dudv = \sigma_S^2\} \qquad (35)$$

and

$$D_n = C_n \qquad (36)$$

where C_n was defined in (4). We assume that the upper and lower bounds and the total powers of the signal and noise processes are known, with $S_L(u,v)$ and $N_L(u,v)$ bounded.

The most robust filter $H_R(u,v)$ is defined as in (5), but now we require the minimization of the MSE rather than maximization of SNR. If $H_R(u,v)$ is the optimum filter for a pair of least favorable PSD's $S_R \epsilon D_s$, $N_R \epsilon D_n$, then $H_R(u,v)$ must satisfy

$$e(S,N;H_R) \leq e(S_R,N_R;H_R) \leq e(S_R,N_R;H) \tag{37}$$

for any pair (S,N) in $D_s \times D_n$ and any linear filter $H(u,v)$. The filter $H_R(u,v)$ and the least favorable pair $S_R(u,v)$, $N_R(u,v)$ will form a saddle-point for the MSE.

The main result is the following theorem:

Theorem 2. For the classes D_s and D_n, the most robust Wiener filter $H_R(u,v)$ exists. It is the optimum filter for the pair of least favorable PSD's defined according to the following:

$$S_R(u,v) = \begin{cases} k_s N_L(u,v) & (u,v)\epsilon\alpha_1(k_s) \\ S_U(u,v) & (u,v)\epsilon\alpha_2(k_s) \\ S_L(u,v) & (u,v)\epsilon\ \bar{\alpha}(k_s) \end{cases} \tag{38}$$

and

$$N_R(u,v) = \begin{cases} \frac{1}{k_n} S_L(u,v) & (u,v)\epsilon\beta_1(k_n) \\ N_U(u,v) & (u,v)\epsilon\beta_2(k_n) \\ N_L(u,v) & (u,v)\epsilon\ \bar{\beta}(k_n) \end{cases} \tag{39}$$

where

$$\alpha_1(k_s) \equiv \{(u,v) \mid S_L(u,v) < k_s N_L(u,v) \le S_U(u,v)\} \quad ,$$

$$\alpha_2(k_s) \equiv \{(u,v) \mid S_L(u,v) \le S_U(u,v) < k_s N_L(u,v)\} \quad ,$$

$$\overline{\alpha}\ (k_s) \equiv \{(u,v) \mid S_L(u,v) \ge k_s N_L(u,v)\} \quad ,$$

$$\beta_1(k_n) \equiv \{(u,v) \mid k_n N_L(u,v) \le S_L(u,v) < k_n N_U(u,v)\} \quad ,$$

$$\beta_2(k_n) \equiv \{(u,v) \mid k_n N_L(u,v) \le k_n N_U(u,v) \le S_L(u,v)\} \quad ,$$

$$\overline{\beta}\ (k_n) \equiv \{(u,v) \mid k_n N_L(u,v) > S_L(u,v)\} \quad , \tag{40}$$

if solutions $k_s < k_n$ exist to the power constraint equations

$$P_S(z) \Big|_{k_s} = \sigma_S^2 \quad , \tag{41a}$$

$$P_N(z) \Big|_{k_n} = \sigma_N^2 \quad , \tag{41b}$$

where

$$P_S(z) = \left(\frac{1}{2\pi}\right)^2 \Big[\iint_{\alpha_1(z)} z N_L(u,v)\,du\,dv + \iint_{\alpha_2(z)} S_U(u,v)\,du\,dv + \iint_{\overline{\alpha}(z)} S_L(u,v)\,du\,dv \Big] \quad , \tag{42a}$$

$$P_N(z) = \left(\frac{1}{2\pi}\right)^2 \Big[\iint_{\beta_1(z)} \frac{1}{z} S_L(u,v)\,du\,dv + \iint_{\beta_2(z)} N_U(u,v)\,du\,dv + \iint_{\overline{\beta}(z)} N_L(u,v)\,du\,dv \Big] \tag{42b}$$

There are two parts to the proof of the theorem, one showing the robustness of the solutions and the other showing existence. The existence proof which depends on finding a solution to (41) or other equations is similar to that giver in [10] and will be omitted. The robustness proof is simpler, and we will outline it here.

In order to show that the above solution satisfies the condition for robustness, we must show (as in [10]) that

$$e(S,N;H_R) - e(S_R,N_R;H_R) \le 0 \tag{43}$$

when $H_R(u,v)$ is the optimum filter for $S_R(u,v)$ and $N_R(u,v)$.

We can consider the two terms, signal and noise, in (33) separately. Define

$$\Delta e_{signal} = (\frac{1}{2\pi})^2 \iint [S(u,v)-S_R(u,v)][\frac{1}{1+S_R/N_R}]^2 \, dudv \tag{44}$$

On $\alpha_1(k_s)$, $S_R/N_R = k_s$ and, on $\beta_1(k_n)$, $S_R/N_R = k_n$ so that $1/(1+S_R/N_R) \le 1/(1+k_s)$. On $\alpha_2(k_s)$, $(S-S_R)$ is non-positive and $S_R/N_R \le k_s$ so that $1/(1+S_R/N_R) \ge 1/(1+k_s)$ and, on $\beta_2(k_n)$, the integrand is non-negative and $1/(1+S_R/N_R) \le 1/(1+k_s)$. Finally, on the rest of the (u,v) plane, $S_R/N_R = S_L/N_L \ge k_s$ and the integrand is non-negative. Thus,

$$\Delta e_{signal} \le (\frac{1}{2\pi})^2 (\frac{1}{1+k_s})^2 \iint [S(u,v)-S_R(u,v)] dudv, \tag{45}$$

and from the power constraint we get

$$\Delta e_{signal} \le 0 \tag{46}$$

In a similar way, the noise term can be shown to be non-positive and robustness is proved.

Notice that although Theorem 2 asserts the existence of a least-favorable pair of PSD's, it only defines them for a particular case when the equations in (41) have solutions with $k_s < k_n$. This also turns out usually to be the case. Otherwise, the least-favorable densities can be obtained directly from the detailed version of the theorem in [10] for the one-dimensional case.

This theorem specifies the least-favorable spectral densities as a pair of spectral densities which tend to be as similar as possible, within the given classes. This is apparent from (38) and (39), and is a reasonable solution from intuitive considerations. In fact, it is possible to prove that in a generalized sense of "distance" between two spectral densities, the solution of Theorem 2 species the minimum distance pair [12].

It should be noted that the result of Theorem 2 can also be applied to another pair of classes E_s and E_n for the signal and noise PSD's, defined by

$$E_s = \{S(u,v) | S(u,v) = (1-\varepsilon_s)S_0(u,v) + \varepsilon_s S_c(u,v) \text{ and}$$

$$\iint S_0(u,v)dudv = \iint S_c(u,v)dudv\} \qquad (47)$$

and similarly for E_n. Here $S_0(u,v)$ is a known nominal signal spectral density, ε_s is an assumed degree of contamination allowed for $S_0(u,v)$, and $S_c(u,v)$ is an otherwise arbitrary spectral density. With $S_L(u,v) = (1-\varepsilon_s)\ S_0(u,v)$ and $S_U(u,v)$ everywhere unbounded in (35), the class D_s becomes the above E_s. Numerical results for such classes in the one-dimensional case given in [10] show the usefulness of these concepts of robust Wiener filtering.

V. CONCLUSION

A design philosophy for filtering signals in noise has been presented, applicable in cases where signal and noise characteristics cannot be modeled precisely. The robust matched and Wiener filters have been derived for two-dimensional discrete-time applications. Such filters can be useful in many two-dimensional signal detection and estimation applications.

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FIGURE CAPTION

Figure 1. Robust filter and least-favorable characteristics considered in Section III.

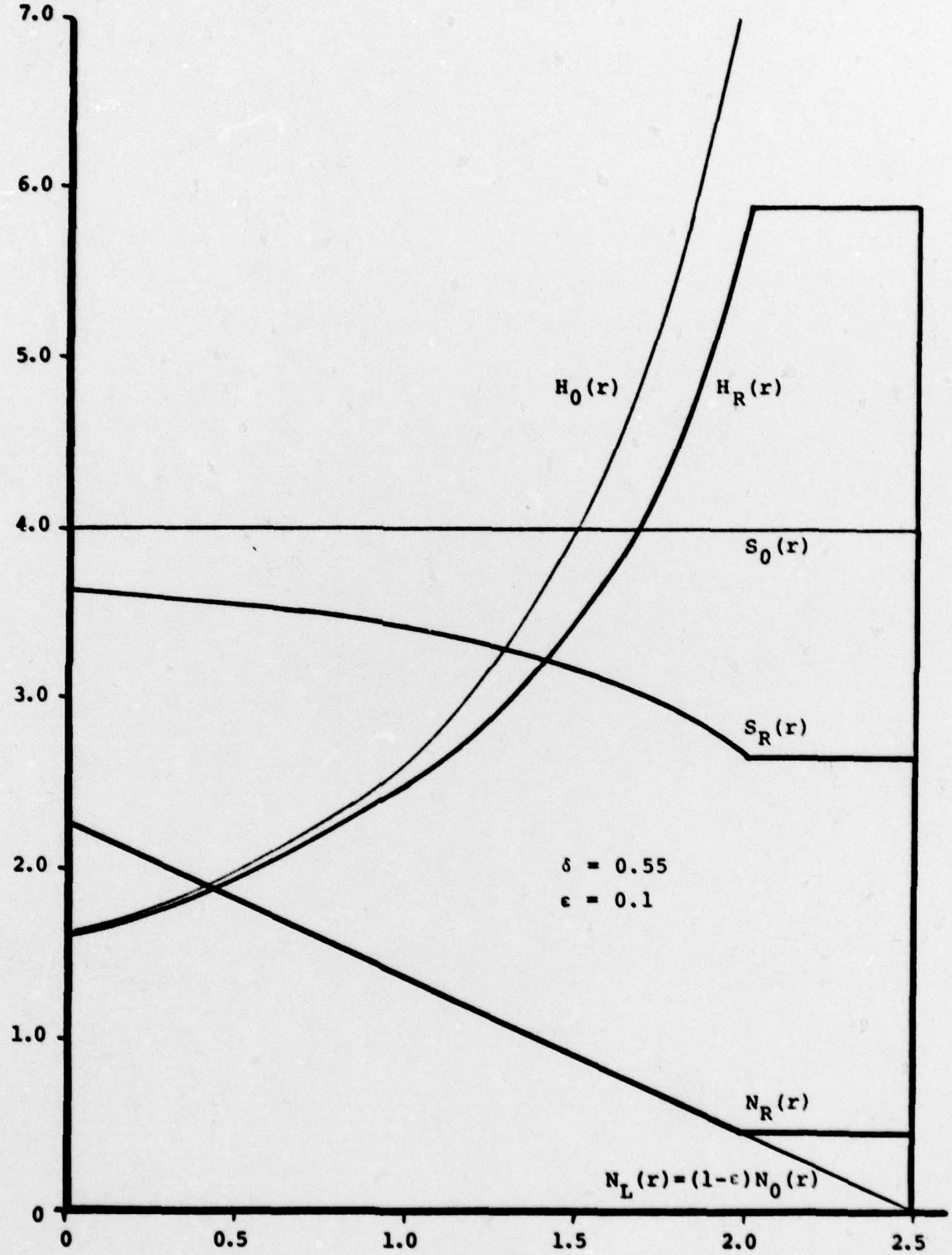

Fig. 1 Kassam/Lim/Cim